Almost all cop-win graphs contain a universal vertex

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(Joint work with Anthony Bonato and Paweł Prałat)

May 2011
The game of cops and robbers
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The cop wins.
The game of cops and robbers

Cops and robbers is a two-player game played on a graph:

1. Cop $C$ chooses vertex.
2. Robber $R$ chooses vertex.
3. $C$ moves along an edge (or passes).
4. $R$ moves along an edge (or passes).
5. Repeat Steps 3 and 4.

$C$ wins if $C$ moves onto $R$. Otherwise, $R$ wins.
The game of cops and robbers

One cop cannot necessarily win...

$C$

$R$

The **cop number** $c(G)$ is the minimum number of cops needed to guarantee that the cops win.
The game of cops and robbers

For a path $P_n$...

For a cycle $C_n$...

For a tree $T$...
The game of cops and robbers

For a path $P_n$...
$c(P_n) = 1$

For a cycle $C_n$...
$c(C_n) = 2$, $n \geq 4$

For a tree $T$...
$c(T) = 1$
The game of cops and robbers

Conjecture [Meyniel ’85]:
For connected \( n \)-vertex graphs \( G \),

\[
c(G) \leq O(\sqrt{n}).
\]

Results

- [Frankl ’87] \( c(G) \leq O \left( \frac{n}{\log n / \log \log n} \right) \)
- [Chiniforooshan ’08] \( c(G) \leq O \left( \frac{n}{\log n} \right) \)
- [Frieze et al ’11+, Lu-Peng ’11+, Scott-Sudakov ’11+] \( c(G) \leq O \left( \frac{n}{2^{(1-o(1))\sqrt{\log_2 n}}} \right) \)
The game of cops and robbers

Conjecture [Meyniel ’85]:
For connected $n$-vertex graphs $G$, $c(G) \leq O(\sqrt{n})$.

Results

- [Prałat ’10] There are graphs with $c(G) \geq d\sqrt{n}$.
The game of cops and robbers

C & R introduction [Nowakowski & Winkler ’83, Quilliot ’78]

C & R on special graphs
  ▶ planar [Aigner & Fromme ’84]
  ▶ product graphs [Neufeld & Nowakowski ’98]
  ▶ infinite [Hahn et al ’02]

C & R with modified rules
  ▶ limited visibility [Isler ’08]
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G is cop-win if \( c(G) = 1 \).

\( C_n \) = the set of cop-win graphs on \( n \) labelled vertices

Questions:

\[ |C_n| = ? \]

\[ |C_n| = (1 + o(1))f(n) \text{ as } n \text{ grows large} \]
Counting cop-win graphs
A vertex is **universal** if it is adjacent to every other vertex.
Counting cop-win graphs

Let $U_n$ be the set of $n$-vertex graphs with a universal vertex.

$$|U_n| = n2^{(n-1)/2} + O(n^22^{(n-2)/2}) = (1 + o(1))n2^{(n-1)/2}$$

So

$$|C_n| \geq |U_n| = (1 + o(1))n2^{(n-1)/2}.$$ 

Surprise! [Bonato, K., Prałat ’11+]:

$$|C_n| = (1 + o(1))n2^{(n-1)/2}.$$ 

$$\frac{|U_n|}{|C_n|} = 1 + o(1).$$

Almost all cop-win graphs contain a universal vertex.
Counting cop-win graphs

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Almost all cop-win graphs contain a universal vertex.
Proving $|C_n| = (1 + o(1))|U_n|$

A vertex $u$ is a corner if $N[u] \subseteq N[v]$ for some vertex $v$. 
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A vertex $u$ is a **corner** if $N[u] \subseteq N[v]$ for some vertex $v$.

**Facts:**

- Every cop-win graph has a corner.
- Deleting a corner from a cop-win graph produces a new cop-win graph.
- [Nowakowski and Winkler ’83, Quilliot ’78]: $G$ is cop-win iff some sequence of deleting corners results in a single vertex.
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For all sequences 

\[ u_1, u_2, \ldots, u_n \]

\[ v_1, v_2, \ldots, v_n \]

count all graphs with \( N[u_i] \subseteq N[v_i] \) that have no universal vertex.

Show that the number of these graphs is small.

Show that the probability of these graphs is small.
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Proving $|C_n| = (1 + o(1))|U_n|$

Random model: each pair of vertices is joined by an edge with probability $1/2$.

- Choose the first $cn$ vertices in this sequence.
- $s = \text{number of distinct } v_i$. Number of choices is $\binom{n}{cn} \binom{n}{s} s^{cn}$.
- Probability that $N[u_i] \subseteq N[v_i]$ is at most $(3/4)^{n-2cn}$.
- These events are independent for at least $s/2$ choices of $i$.

Also condition on degrees of $v_i$. 
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Also condition on degrees of $v_i$. 
Let $C_n^{(k)}$ be the set of all $n$-vertex $k$-cop-win graphs. $|C_n^{(k)}| = ?$

Conjecture: almost all $k$-cop-win graphs contain a $k$-vertex dominating set.

Corollary:

$$|C_n^{(k)}| = 2^{o(n)}(2^k - 1)^{n-k}2^{\binom{n-k}{2}}.$$ 

[Aigner-Fromme ’84] $c(G) \leq 3$ for $G$ planar.

Question: Which of these require 1, 2, or 3 cops?
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