Open-Market Operations, Asset Distributions, and Endogenous Market Segmentation

Babak Mahmoudi*

November 5, 2012

Abstract

This paper investigates the long-run effects of open-market operations on the distributions of assets and prices in the economy. It offers a theoretical framework to incorporate multiple asset holdings in a tractable heterogeneous-agent model, in which the central bank implements policies by changing the supply of nominal bond and money. This model features competitive search, which produces distributions of money and bond holdings as well as price dispersion among submarkets. At a high enough bond supply, the equilibrium shows segmentation in the asset market; only households with good income shocks participate in the bond market. When deciding whether to participate in the asset market households compare liquidity services provided by money with returns on bond. Segmentation in the asset market is generated endogenously without assuming any rigidities or frictions in the asset market. In an equilibrium with a segmented asset market open-market operations affect households’ participation decision and, therefore, have real effects on the distribution of assets and prices in the economy. Numerical exercise shows that the central bank can improve welfare by purchasing bonds and supplying money.

JEL Classification Numbers: E0, E4, E5

Keywords: Open-Market Operation, Distributional Effects, Segmented Asset Market, Heterogeneous Agents, Competitive Search

*Department of Economics, Queen’s University, Kingston, Ontario, Canada, K7L 3N6. Email: mahmoudi@econ.queensu.ca. I thank my supervisors, Hongfei Sun and Allen Head, for their continuous support. I am also grateful for the valuable comments and suggestions of Huw Lloyd-Ellis, Thorsten Koepppl, Victor Rios-Rull, Jonathan Chiu, Marco Cozzi, and participants in the Summer Workshop on Money, Banking, Payments and Finance at the Federal Reserve Bank of Chicago, the Canadian Economics Association Conference, the CIREQ Ph.D. Student Conference, the Royal Economic Society Annual Conference, and the Midwest Economics Association Annual Meeting. This research has been supported by the Ontario Graduate Scholarship (OGS) awards program and the John Deutsch Institute.
1 Introduction

What are the long-run effects of open-market operations on the distribution of assets and prices in the economy? Why do some people participate in the market for interest-bearing assets and hold positive portfolios of these assets, while others do not participate in the asset market and only hold money? In order to answer these questions, I construct a model of monetary economy with heterogeneous agents, in which the central bank implements policies by changing the supply of nominal bonds and money. I show that a segmented asset market arises under a specific parameters set; households with high income participate in the bond market and hold positive portfolio of bond and money, while low income households do not participate in the bond market and only keep money for transaction purposes. When deciding whether to participate in the asset market households compare liquidity services provided by money with returns on bond. In an equilibrium with a segmented asset market, open-market operations affect the participation decisions of the households and, therefore, has real effects on the distribution of assets and prices in the economy.

Distribution of asset holding and segmentation in the asset market is well documented. Some agents participate the market for interest bearing assets and hold positive portfolios of different assets while others do not participate the asset market. Explaining these facts requires a heterogeneous agent model in which households choose to hold different portfolios of asset holdings. In this paper, households have different preferences towards labor supply, and they experience different matching shocks. Heterogeneity in preferences towards labor supply and idiosyncratic matching shocks allow me to generate an equilibrium distribution of asset holding among different households and a distribution of prices among different markets.

A branch of literature uses asset market segmentation to explain persistence responses to monetary shocks observed in the data. In these models with segmented asset markets, only the fraction of agents who are active in the asset markets immediately receive the monetary shocks. As a result, it would take time for the monetary shocks to affect other agents in the economy. This literature explains real effect of money injection and open-market operations with the generated segmentation in the asset market. These models use two ways to generate the segmented asset market: limited participation models that assume

---

17.7% of the surveyed U.S. households did not have access to banking products and services in 2009, and at least 71% of these unbanked U.S. households earned less than $30,000 in a year. 26.5% of U.S. households did not have any savings in a bank account, moreover they did not hold any financial assets similar to a bank account. Source: FDIC (2009) and The Panel Study of Income Dynamics (PSID). According to the Survey of Consumer Finances (2010), 92.5% of households had access to transaction accounts and 12% held saving bonds.

2For an overview of this literature see Edmond and Weill (2008)
only certain agents attend the asset market and models that assume agents must pay a fixed cost to enter the asset market or to transfer assets between asset market and goods market. In a cash-in-advance framework, Grossman and Weiss (1983) assume that only a fixed fraction of the population can withdraw funds from banks each period. In Alvarez et al. (2000), agents must pay a fixed cost to transfer money between the asset market and the goods market. In a similar fashion, Khan and Thomas (2010) assumes agents pay idiosyncratic fixed costs to transfer wealth between interest-bearing assets and money. Chiu (2007) assumes that agents pay a fixed cost to attend the asset market and they choose the timing of money transfers. In a micro-founded monetary framework, Williamson (2008) links the asset market segmentation to the goods market segmentation.

In this paper, I generate segmentation in the asset market without assuming any rigidities and frictions in the asset market. All of the agents can attend the asset market every period, and there is no transaction cost or any other frictions that prohibit agents from trading in the asset market. Segmentation in the asset market is generated endogenously. When deciding whether to participate in the asset market, households compare liquidity services provided by money with return on bond. Agents hold different amounts of assets, and some agents choose to hold no bond in their asset portfolio. In a segmented asset market, the return on bond is not high enough to attract all of the households to the asset market. Here, the real and welfare effects of open-market operations and money injections are not caused by the segmentation in the asset market. However, open-market operations have real effects on the distributions of assets and prices when the markets are segmented. In a segmented asset market, agents at the participation margin of trading assets may change their decision with a marginal change in the bond supply. Numerical exercises show that the central bank can improve welfare by purchasing bonds and supplying money. The policy of open-market purchase of bond is most effective when the asset market is segmented. This policy would increase the participation rate in the asset market and help households smooth consumption. By participating in the asset market, households are able to better insure themselves against idiosyncratic income shocks. Moreover, in a segmented asset market open-market purchase of bond decreases both the intensive and extensive margins of trade in the decentralized market. The results are robust to exogenous segmentation in the asset market.

Following Wallace (1981), a branch of literature uses a Modigliani-Miller argument to show that the size and the composition of the central bank balance sheet and thus open-market operations do not have any real effect on the economy. In these models, assets are perfectly substitutable in terms of liquidity services. Open-market operations do not change the liquidity characteristics of households’ asset portfolio. Shi (2008) and Williamson (2011)

---

3Similarly Alvarez et al. (2001) assumes only a fixed fraction of agents attend the asset market.
assume assets other than money provide partial liquidity services. In these models, open-market operations change the overall liquidity in the economy. Because of partial liquidity, government bonds are not perfect substitutes for money and a Modigliani-Miller argument does not hold. The same logic is applied in this paper. Agents can only trade with money and government bond is completely illiquid in the market for goods. Government bond is an imperfect substitute for money, thus open-market operations can have real effects on the economy. Here, households with good shocks use nominal bond to smooth their consumption over time and illiquidity of bond is important for this purpose. In a model with liquid bond, since bond and money are perfectly substitutable, households are indifferent between holding bond and money. They cannot use bond to smooth consumption. Kocherlakota (2003) uses a similar argument and shows that in a centralized market, agents use illiquid bonds to smooth consumption intertemporally.

This paper is related to the literature on the distribution of money and assets in the economy. In a search model of monetary economy with bargaining, after each round of trading there would be agents that have been matched and have succeeded in trade and agents that have not traded. This would generate an evolving distribution of asset holding among agents, which is a state variable and makes the model intractable. Camera and Corbae (1999) generate distribution of asset holdings among agents and price dispersion in equilibrium in a framework based on Kiyotaki and Wright (1989). The evolving distribution of asset holding makes their model highly intractable for policy analysis. A large section of the monetary literature avoids the distribution of asset holding by simplifying assumptions. Lucas (1990) and Shi (1995) assume a large household structure and with this insurance mechanism, agents within a household share consumption and asset holdings after each round of trade. The sharing mechanism collapses the distribution of asset holding to a single point. Lagos and Wright (2005) assume a quasi-linear preference structure for the agents along with one round of centralized trading. These assumptions make the distribution of money holding degenerate and the model highly tractable. By using competitive search in the decentralized market for goods Menzio et al. (2011) are able to make the distribution of money holding non-degenerate. Sun (2012) puts Menzio et al. (2011) in a Lagos-Wright framework and by using a household structure, the model becomes more tractable for studying the effects of different fiscal and monetary policies. Models in Sun (2012) and Menzio et al. (2011) are block recursive; the household’s problem can be solved without involving the endogenous distribution of asset holding.

My paper closely follows Sun (2012) in using competitive search in the decentralized

---

4 Zhu (2003), Zhu (2005) and Green and Zhou (1998) use similar approach and have distribution of asset holding and price dispersion as an equilibrium object
market together with a centralized market to adjust asset balances. Households and firms trade goods in markets with and without frictions. The frictional markets are characterized by competitive search, where households face a trade-off between higher matching probability and better terms of trade. Competitive search in the goods market makes the model highly tractable.\(^5\) Households with different idiosyncratic labor cost shocks choose to hold different amounts of assets. Search is directed in the sense that households with different portfolios of asset holding have different preferences towards matching probability and terms of trade and choose different submarkets. Agents with a high income shock choose a submarket with high price and higher matching probability. Despite a nontrivial distribution of money and bond across agents, competitive search in the frictional markets makes this model highly tractable.

Unlike models with bargaining and due to the competitive nature of the frictional goods market, the distribution of households across asset holdings does not directly affect the firms’ cost/benefit of opening a shop in a submarket. Households’ decisions do not affect matching probabilities and terms of trade in the frictional goods market. Households take the specification of the submarkets as given and choose which submarket to participate. Households only need to know the prices in the economy, and these prices contain all of the information about the distributions in the economy. Hence, the equilibrium is partially block recursive.\(^6\) Households’ decisions do not directly depend on the distribution of asset holding in the economy.

The rest of the paper is organized as follows. In Section 2, I develop the model environment and characterize value and policy functions. Section 3 defines and characterizes the stationary equilibrium. Section 4 presents the computational algorithm and the results of numerical example. In Section 5, I introduce exogenously segmented asset market to the model. Section 6 concludes the paper.

## 2 Model environment

Time is discrete, and each period consists of four subperiods; asset market, labor market, frictionless market and frictional market. The economy is populated by measure 1 of ex-ante

\(^5\)Aside from tractability, comparing to random search, competitive search is closer to the real world. e.g. as Howitt (2005) states:”In contrast to what happens in search models, exchanges in actual market economies are organized by specialist traders, who mitigate search costs by providing facilities that are easy to locate. Thus, when people wish to buy shoes they go to a shoe store; when hungry they go to a grocer; when desiring to sell their labor services they go to firms known to offer employment. Few people would think of planning their economic lives on the basis of random encounters with nonspecialists…”

\(^6\)Unlike a block recursive equilibrium (Shi (2009), Menzio and Shi (2010) in labor, and Menzio et al. (2011) and Sun (2012) in monetary economics), here distributions affect households’ decision through prices
homogeneous households. Each household consists of a worker and a buyer. There is a general good that can be produced and consumed by all of the households. There are also at least three types of special goods. Each household is specialized in the production and consumption of one of the special goods, and there is no double coincidence of wants. Because of the specialized structure of households and no-double-coincidence-of-wants assumption, a medium of exchange is necessary in this environment. The utility function of the household is

\[ U_h(y, q, l) = U(y) + u(q) - \theta l \]

where \( y \) is the consumption of the general good, \( q \) is the consumption of the special goods, and \( l \) is the labor supply in a period of time. The parameter \( \theta \in [\underline{\theta}, \bar{\theta}] \) is the random disutility of labor. It is iid across households and time, and it is drawn from the probability distribution \( F(\theta) \) at the beginning of each period. \( \theta \) captures the heterogeneity of households. \( U() \) and \( u() \) are continuous and twice differentiable. \( u' > 0, U' > 0; u'' < 0, U'' < 0; u(0) = U(0) = u'(\infty) = U'(\infty) = 0; \) and \( u'(0) \) and \( U'(0) \) are large and finite. Goods are divisible and perishable. There are two fiat objects in the economy: money and nominal bond. They are supplied by the central bank. Nominal bond is supplied in a centralized market after the utility shocks has been realized. Agents redeem each unit of bond from last period for 1 unit of money at the beginning of each period. Bond can be costlessly counterfeited by all of households, and they cannot recognize original bond that is printed by the central bank and fake bonds that are printed by other households. As a result, nobody accepts bond as a medium of exchange and households cannot trade with bond.

Agents can trade the general good in a perfectly competitive market, called frictionless market. There are search frictions in the market for special goods. Following Peters (1991) and Moen (1997), I assume a competitive search environment where agents choose to search in submarkets indexed by terms of trade and matching probability. Agents are randomly matched, and only matched agents can trade goods. There is a measure one of competitive firms, who hire workers from the households at the beginning of a period in a competitive labor market. Firms pay hired workers by issuing IOUs. These IOUs can be used to trade goods and are redeemed at the end of the period. Households own equal shares in these firms. Firms need labor for production of the general good and one type of special goods. These firms are destroyed at the end of each period and new firms are formed in the second period.

---

\( ^7 \)Bond liquidity services have been discussed in the literature e.g. Shi (2008), Mahmoudi (2011) and Kocherlakota (2003).

\( ^8 \)Because of the large structure of firms, they do not face unpredictable matching shocks, and there is no commitment problem in redeeming IOUs.
subperiod of each period\(^9\). I assume free entry for the firms, therefore the number of firms follows a zero profit condition.

In the frictional market there exists a continuum of submarkets that have specific characteristics in terms of trade and matching probabilities. Firms choose the measure of shops to operate in each submarket. There is free entry in these submarkets. The fixed cost of operating a shop in a submarket is \(k > 0\) units of labor. In producing \(q\) units of special goods, firms incur \(\psi(q)\) units of labor in production cost. Where \(\psi()\) is twice continuously differentiable and \(\psi' > 0, \psi'' > 0\) and \(\psi(0) = 0\).

Trading in these submarkets is characterized by competitive search. Each submarket is a particular set of terms of trade (\(q\): amount of special goods and \(x\): money to be paid) and matching probabilities (\(b\): matching probability for buyers and \(e\): matching probability for sellers/shops). Firms and households take terms of trade and matching frictions as given and decide which submarket to participate. In each submarket buyers and shops randomly match according to the respective matching probability. Households and firms decide which submarket to enter, therefore matching probabilities are a function of terms of trade \((x, q)\). Each submarket can be indexed by the respective terms of trade. I assume that matching probability is characterized by a constant return to scale matching function \((e = \mu(b))\), which has the standard characteristics of a matching function.

At the beginning of the period, government prints money at rate \(\gamma\), redeems last period nominal bonds \((A_{-1})\) for 1 unit of money, and issues and sells bonds \((A)\) for the current period at nominal price \(s\) and balances budget by a lump sum tax/transfers \((T)\). The asset market is a competitive market and households take bond price \((s)\) as given.

I study the steady state equilibrium, and I will use labor as the numeraire of the model. Figure 1 shows the timing of events.

### 2.1 Firms’ decision

Firms have access to a linear production technology. For each unit of labor input they produce one unit of output. Firms decide how much to produce in the frictionless market \((Y)\) and the measure of shops in each submarket \((dN(x, q))\). They sell the produced general good at the given market price \(P\). In each submarket the matching probability for each shop is \(e(x, q)\). Shops sell the produced special goods to matched buyers at price \(x\). In the production process, firms incur \(k\) units of labor in fixed cost, and \(\psi(q)\) units of labor in variable costs. Firms maximize the following profit function

\[\text{Maximize } \pi = \sum_{x, q} (P \cdot q - k - \psi(q)) \cdot e(x, q)\]

---

\(^9\)With this assumption, there is no need to keep track of firms’ asset holdings
\[ \pi = \max_Y \{ PY - Y \} + \max_{dN(x,q)} \int \{ e(x,q) x - [k + e(x,q)\psi(q)] \} dN(x,q) \]  

Figure 1: Timing

If the expected profit in a submarket is strictly positive, firms will choose \( dN(x,q) = \infty \). If the expected profit is strictly negative firms will choose \( dN(x,q) = 0 \). Therefore, the optimal \( dN(x,q) \) satisfies the following inequalities with complementary slackness

\[ e(x,q)[x - \psi(q)] \leq k \quad dN(x,q) \geq 0 \]  

As is standard in the competitive search literature, I assume that the profit maximizing condition holds for the submarkets that are not visited by any buyers and firms. For all submarkets where \( k < x - \psi(q) \) we have

\[ e(x,q)[x - \psi(q)] = k \]

\[ dN(x,q) = 0 \]

For the submarkets where \( k \geq x - \psi(q) \) we have \( dN(x,q) = 0 \), and I assume \( e = 1 \) and \( b = 0 \). I can write these two cases as

\[ e(x,q) = \begin{cases} \frac{k}{x - \psi(q)} & k \leq x - \psi(q) \\ 1 & k > x - \psi(q) \end{cases} \]  

Note that the matching probabilities do not depend on the distributions in the economy.
This property of the frictional market simplifies the households problems, and we can write households’ matching probabilities as a function of terms of trade \((x, q)\).

### 2.2 Households’ decision

#### 2.2.1 Decision in the frictionless market

In the beginning of each period a centralized asset market opens. Households redeem each unit of their nominal bonds from previous period for 1 unit of money. Government prints and injects money at rate \(\gamma\). Government supplies one period nominal bonds in a centralized market at the competitive price \(s\). The asset market closes until the next period.

Let \(W(m, a_{-1}, \theta)\) be the value function of a representative household at the beginning of a period. The representative household holds a portfolio composed of \(m\) units of money and \(a_{-1}\) units of nominal bonds in units of labor at the beginning of the period. I take labor as the numeraire. Let \(w\) be the normalized wage rate, which is the nominal wage rate divided by the money stock \((M)\). The nominal wage rate associated with real balance \(m\) is \(wMm\).

Given the prices \((p, s)\) and transfers \((T)\), the household decides on how much to consume in the frictionless market \((y \geq 0)\), labor supply \((l \geq 0)\), money balances for transaction purposes\((z)\), money balances for precautionary saving \((h)\) and bond holdings at the beginning of the following period \((a)\). Let \(V(z, h, a)\) be the value of the representative household at the start of the next subperiod (frictional market). She chooses an asset portfolio consisting of the money needed for transaction purposes \((z)\), precautionary savings \((h)\) and bond holding \((a)\) for the frictional market. In order to purchase nominal bonds, one has to pay the nominal price \(s\) this period to receive the nominal return 1 in the following period. In order to have a real return \(a\) in the following period, we need to pay \(s\gamma a\) in terms of labor units. The households solves the following optimization problem subject to a standard budget constraint.

\[
W(m, a_{-1}, \theta) = \max_{y, l, z, h, a} U(y) - \theta l + V(z, h, a)
\]

\[
\text{st. } py + z + h + s\gamma a \leq m + a_{-1} + l + T
\]

Let's assume that \(V(z, h, a)\) is differentiable and the choice of \(l\) is an interior solution (I will prove these later). As \(U\) is positively sloped the budget constraint is binding. I use the binding budget constraint to eliminate \(l\) from the optimization problem. Using the equilibrium condition \(p = 1\), the value function of the representative household can be written as
\[ W(m, a_{-1}, \theta) = \theta(m + T + a_{-1}) + \max \{ U(y) - \theta py \} + \max \{ -\theta(z + s\gamma a + h) + V(z, h, a) \} \]

The above expression is linear in the household’s portfolio of asset holding at the beginning of the period \((m, a_{-1})\). As I will show later, this linearity will simplify the problem of the household in the decentralized market for goods. Furthermore, the households’ choice of asset holding for the following subperiod \((z, h, a)\) is independent of current subperiods’ asset holding \((m, a_{-1})\).\(^{10}\)

The optimal choices of \(y\) must satisfy

\[ U'(y) = \theta \] (4)

In the above equation, I have used the equilibrium condition \(p = 1\). Similarly \(z, h\) and \(a\) satisfy

\[
\begin{align*}
V_z(z, h, a) &\begin{cases} 
\leq \theta & z \geq 0 \\
\geq \theta & z \leq m - s\gamma a - h 
\end{cases} \\
V_h(z, h, a) &\begin{cases} 
\leq \theta & h \geq 0 \\
\geq \theta & h \leq m - s\gamma a - z 
\end{cases} \\
V_a(z, h, a) &\begin{cases} 
\leq \theta s\gamma & a \geq 0 \\
\geq \theta s\gamma & sa\gamma \leq m - z - h 
\end{cases}
\end{align*}
\] (5, 6, 7)

Where the inequalities hold with complimentary slackness. \(m\) is the maximum amount of money that households can hold in terms of labor units. Clearly households money balance \((m)\), and bond holding \((a_{-1})\) does not affect the choices of \(y, z, h\) and \(a\). This is an important property of households’ policy function. The households’ decisions are independent of their current portfolio of asset holdings. As a result, I can write policy functions as functions of households’ type \((\theta)\). Using the optimization problem of the household, I can write the value function as a linear function of \(m\) and \(a_{-1}\)

\[ W(m, a_{-1}, \theta) = W(0, 0, \theta) + \theta m + \theta a_{-1} \] (8)

Where

\[ W(0, 0, \theta) = U(y(\theta)) - \theta y(\theta) + V(z(\theta), h(\theta), a(\theta)) - \theta(z(\theta) + h(\theta) + s\gamma a(\theta)) \] (9)

\(^{10}\)The quasi-linear preference structure allows me to remove the wealth effects.
It is clear that the value function is continuous and differentiable. The following lemma summarizes these findings.

**Lemma 1.** The value function $W(m, a_{-1}, \theta)$ is continuous and differentiable in $(m, a_{-1}, \theta)$. It is also affine in $m$ and $a_{-1}$.

Lemma 1 shows the standard linearity property that is shared by frameworks based on Lagos and Wright (2005). I can use this property to simplify household’ decision in the frictional market.

### 2.2.2 Decision in the frictional market

The representative household’s decision in the frictional market is similar to Sun (2012). The representative household chooses which submarket to participate. As I can index the submarkets by the respective terms of trade, the household chooses the terms of trade ($x$ and $q$) to maximize expected value of attending the respective submarket. In choosing which submarket to participate, households are constrained by their amount of money holding ($x \leq z$).\(^\text{11}\) In a submarket the household matches with probability $b(x, q)$, and trades according to the stated terms of trade. The matching probability comes from the firms’ decision problem. In each match the representative household spends $x$ amount of money, and consumes $q$ amount of special good. With probability $1 - b(x, q)$ there is no match and the representative household exits the frictional market with the starting portfolio of assets. As is standard in the search and matching literature, I assume $b(x, q)$ is nonincreasing. The representative household solves the following optimization problem

$$v(z, h, a) = \max_{x \leq z, q} b(x, q) \left[ u(q) + \beta E[W\left(\frac{z - x + h}{\gamma}, a, \theta\right)] \right] + [1 - b(x, q)]\beta E \left[W\left(\frac{z + h}{\gamma}, a, \theta\right)\right]$$

(10)

Using the linearity of $W(.)$ (8) and firms’ optimization problem (3), I can eliminate $q$ from the above expression. The households’ problem becomes

\(^{11}\)Because households are committed to posted terms of trade they cannot choose a submarket in which they cannot afford to trade.
\[ v(z, h, a) = \max_{x \leq z, b} \left\{ b \left[ u\left(\psi^{-1}(x - \frac{k}{\mu(b)})\right) - \beta E(\theta) \frac{x}{\gamma} \right] + \beta E \left[ W\left(\frac{z + h}{\gamma}, a, \theta\right) \right] \right\} \]  
\tag{11}

The optimal choices of \( x \) and \( b \) satisfy the following first-order conditions

\[ \frac{u'}{\psi'} \left( \psi^{-1}(x - \frac{k}{\mu(b)}) \right) - \frac{\beta E(\theta)}{\gamma} \geq 0, \quad x \leq z \]  
\tag{12}

\[ u \left( \psi^{-1}(x - \frac{k}{\mu(b)}) \right) - \frac{\beta E(\theta) x}{\gamma} + \frac{u' \left( \psi^{-1}(x - \frac{k}{\mu(b)}) \right) \frac{kb\mu'(b)}{[\mu(b)]^2}}{\psi'} \left( \psi^{-1}(x - \frac{k}{\mu(b)}) \right) \leq 0, \quad b \geq 0 \]  
\tag{13}

where the two sets of inequalities hold with complementary slackness. Note that \( b = 1 \) cannot be an equilibrium outcome\(^{12}\).

For \( b(z) = 0 \), I assume \( x(z) = z \). Define \( \Phi(q) = \frac{u'(q)}{\psi'(q)} \). As is shown in Sun (2012), without loss of generality, I can focus on the case \( x(z) = z \). Similar to Sun (2012), households do not need to hold more money than they want to spend. If the following condition holds\(^{13}\)

\[ u \left( \phi^{-1} \left( \frac{\beta E(\theta)}{\gamma} \right) \right) - \frac{\beta E(\theta)}{\gamma} \left( \psi \left( \phi^{-1} \left( \frac{\beta E(\theta)}{\gamma} \right) \right) + k \right) > 0 \]  
\tag{14}

The household’s problem becomes

\[ B(z) + \beta E \left[ W\left(\frac{z + h}{\gamma}, a, \theta\right) \right] \]  
\tag{15}

\(^{12}\)\( b = 1 \) implies \( e = 0 \), \( dN(z, q) = \infty \), and positive profits for the firms. This violates free firms’ entry condition.

\(^{13}\)Let’s assume for \( b(z) > x(z) < z \). Then 11 is independent of \( z \). 13 holds with equality and can be written as:

\[ q^* = \Phi^{-1} \left[ \frac{\beta E(\theta)}{\gamma} \right] \]

Given \( q^* \), 13 can be written as:

\[ u(q^*) - \frac{\beta E(\theta)}{\gamma} \left[ \psi(q^*) + \frac{k}{\mu(b^*)} \right] + \frac{u'(q^*)}{\psi'(q^*)} \frac{kb\mu'(b^*)}{[\mu(b^*)]^2} = 0 \]

The left-hand side of the above equation is strictly increasing in \( b^* \), and \( b^* \) exists and is unique if \( \frac{E(\theta)}{\gamma} \) satisfies:

\[ u(q^*) - \frac{\beta E(\theta)}{\gamma} \left[ \psi(q^*) + k \right] > 0 \]

For all \( z < x^* = \psi(q^*) + \frac{k}{\mu(b^*)} \), \( x(z) = z \). For \( z \geq x^* \), \( x(z) = x^* \).
where

\[ B(z) = \max_{b \in [0,1]} \left[ u\left(\psi^{-1}(z - \frac{k}{\mu(b)}) - \beta \frac{z}{\gamma} E(\theta)\right) \right] \]  

(16)

The value function \( B(z) \) may not be concave in \( z \). Furthermore, equation 16 is the product of the choice variable \( b \) and a function of \( b \) and this product may not be concave. Following Menzio et al. (2011) and Sun (2012), I introduce lotteries to make the households’ value function concave\(^\text{14}\). A lottery is a choice of probabilities \((\pi_1, \pi_2)\) and respective payments \((L_1, L_2)\) that solves the following problem

\[ \tilde{V}(z) = \max_{L_1, L_2, \pi_1, \pi_2} \left[ \pi_1 B(L_1) + \pi_2 B(L_2) \right] \]  

(17)

Subject to

\[ \pi_1 L_1 + \pi_2 L_2 = z; \quad L_2 \geq L_1 \geq 0 \]

\[ \pi_1 + \pi_2 = 1; \quad \pi_i \in [0,1] \]

Note that the agent’s policy functions for the lottery choices are: \( L_{i \in \{1,2\}}(z) \) and \( \pi_{i \in \{1,2\}}(z) \). \( \tilde{V}(z) \) is the households’ value function after playing the lottery. After playing this lottery, the value function of the household becomes concave.

### 2.3 Government

Government imposes policy by either changing the inflation rate (\( \gamma \)) or changing the relative supply of bond (\( \lambda \)). I assume that the government runs a balanced budget at each period. Let’s define

\[ \lambda = \frac{A_{-1}}{M} \]

as the ratio of stock of bond to stock of money in the economy. \( \lambda \) shows the composition of the central bank’s balance sheet. A temporary jump in \( \lambda \) indicates that the central bank has issued more short-term debt, and the composition of its balance sheet has shifted to short-term financing of the government transfers. The total real transfer that a household receives (\( T \)) is the sum of transfers from printing money (seigniorage) and the transfers received from bond market\(^\text{15}\).

---

\(^{14}\)Numerical exercise in section 4 shows that households play lotteries only when they have very low real balances and this does not happen in equilibrium.

\(^{15}\)Note that because of the quasi-linear structure of households’ utility function, government transfers can be interpreted as a public good.
\[ T = \frac{\gamma - 1}{w^\gamma} + \frac{sA - A_{-1}}{wM'} \]  

(18)

### 2.4 Properties of value and policy functions

Here, I characterize policy functions and value functions. As shown in the previous section, the choice of bond holdings and bond prices does not directly affect households’ decision in the frictional market. Solution to firms’ problem (3) shows that the matching probabilities do not depend on the distributions in the economy. **Sun (2012)** discusses fiscal policy in a framework similar to the one used here. Fiscal policy variables do not directly affect households’ decision in the frictional market. As a result, the properties of value functions and the households’ choice of which submarket to search \((x,q)\) are the same as in **Sun (2012)**. Let’s define \(\hat{z}\) as the maximum value of real balance \((z)\) that the equation 14 holds. The following lemma shows the properties of the value functions and policy functions:

**Lemma 2.** The following statements about the value functions and policy functions are true

1. The value function \(B(z)\) is continuous and increasing in \(z \in [0, \hat{z}]\)

2. The value function \(\tilde{V}(z)\) is continuous, differentiable, increasing and concave in \(z \in [0, \hat{z}]\).

3. For \(z\) such that \(b(z) = 0\), the value function \(B(z) = 0\) and the choice of \(q\) is irrelevant.

4. If and only if there exists a \(q > 0\) that satisfies

\[
 u(q) - \frac{\beta E(\theta)}{\gamma} [\psi(q) + k] > 0
\]

There exists a \(z > 0\) such that \(b(z) > 0\)

5. For \(z\) such that \(b(z) > 0\), the value function \(B(z)\) is differentiable, \(B(z) > 0\) and \(B'(z) > 0\).

6. \(b(z)\) and \(q(z)\) are unique and

\[
 b'(z) > 0 \\
 q'(z) > 0
\]

7. \(b(z)\) solves

\[
 \max_{b \in [0,1]} \left\{ u(q(z)) - \frac{\beta E(\theta)z}{\gamma} + \frac{u'(q(z)) k b \mu'(b)}{\psi'(q(z)) \left[\mu(b)\right]^2} \right\} 
\]

(19)
where:
\[ q(z) = \psi^{-1}\left(z - \frac{k}{\mu(b(z))}\right) \] (20)

8. \( b(z) \) strictly decreases with \( E(\theta) \), and \( q(z) \) strictly increases in \( E(\theta) \)

9. There exists \( z_1 > k \) such that \( b(z) = 0 \) for all \( z \in [0, z_1] \) and \( b(z) > 0 \) for all \( z \in (z_1, \hat{z}] \)

10. There exists \( z_0 > z_1 \) such that a household with \( z < z_0 \) will play the lottery with the prize \( z_0 \).

Since the choices of bond holdings and bond prices do not directly affect households’ decision in the frictional market the proof of lemma 2 is exactly similar to lemma 2 in Sun (2012). Lemma 2 summarizes the characteristics of the value functions. According to part 6 households with higher money balances choose to trade in submarkets with higher matching probabilities and higher terms of trade. They sort themselves in different submarkets according to their money holdings. A household with a higher money balance has lower marginal value for money. She wants to get rid of a high amount of money in a short period of time and therefore chooses a submarket with high price and high matching probability.

Equations 8, 10, 16 and 17 give

\[ V(z, h, a) = \tilde{V}(z) + \beta E[W(z + \frac{h}{\gamma}, a, \theta)] \]
\[ = \tilde{V}(z) + \beta E[W(0, 0, \theta)] + \frac{\beta E(\theta)z}{\gamma} + \frac{\beta E(\theta)h}{\gamma} + \beta E(\theta)a \] (21)

Equation 21 shows that \( V(z, h, a) \) is linear in \( a \) and \( h \), and the slopes are

\[ V_a(z, h, a) = \beta E(\theta) \] (22)
\[ V_h(z, h, a) = \frac{\beta E(\theta)}{\gamma} \] (23)

Using lemma 2, equations 22 and 23 and policy functions 26 and 25, I can conclude the following lemma

**Lemma 3.** The value function \( V \) is continuous and differentiable in \((z, h, a)\). \( V(z, h, a) \) is increasing and concave in \( z \in [0, \hat{z}] \). \( V(z, h, a) \geq \beta E[W(0, 0, \theta)] > 0 \) for all \( z \).

Continuity and differentiability of \( V \) with respect to precautionary savings \((h)\) and bond holdings \((a)\) is trivially concluded from the linearity condition \((21, 22, 23)\). \( V \) is increasing
and concave in \( z \in [0, \hat{z}] \), because equation 21 can be differentiated as

\[
\frac{\partial V(z, h, a)}{\partial z} = \frac{d\tilde{V}(z)}{dz} + \frac{\beta E(\theta)}{\gamma}
\]  

(24)

and lemma 2 shows that \( \tilde{V}(z) \) is increasing and concave in \( z \).

Using conditions 6, 7, 22 and 23, I can write the household’s choice of bond holding and precautionary saving in money as follows

\[
\begin{align*}
\{ & h(\theta) \geq 0 &, \quad \theta \geq \frac{\beta E(\theta)}{\gamma} \\
& h(\theta) \leq \overline{m} - z(\theta) - s\gamma a(\theta) &, \quad \theta \leq \frac{\beta E(\theta)}{s\gamma} \\
\} \quad & a(\theta) \geq 0 &, \quad \theta \geq \frac{\beta E(\theta)}{s\gamma} \\
& s\gamma a(\theta) \leq \overline{m} - z(\theta) - h(\theta) &, \quad \theta \leq \frac{\beta E(\theta)}{s\gamma}
\end{align*}
\]  

(25)

where the inequalities hold with complementary slackness. Using equations 21, 26, 25 and 24, I can characterized the policy functions of the households with respect to asset holdings and labor supply in Lemma 4

**Lemma 4.** \( a(\theta), h(\theta), z(\theta) \) and \( l(m, a_{-1}, \theta) \) follow the following rules:

**Case I: Negative nominal interest rate (s ≥ 1)**

\[
\begin{align*}
\{ & \theta < \frac{\beta E(\theta)}{\gamma} \\
& h(\theta) = \overline{m} - z(\theta) \\
& a(\theta) = 0 \\
& l(m, a_{-1}, \theta) = py(\theta) + \overline{m} - a_{-1} - T \\
& V_z = \tilde{V}_z(z) \\
\} \quad & h(\theta) = 0 \\
& a(\theta) = 0 \\
& l(m, a_{-1}, \theta) = py(\theta) - m - a_{-1} - T \\
& V_z = \tilde{V}_z(z) + \frac{\beta E(\theta)}{\gamma}
\end{align*}
\]  

(27)

**Case II: Positive nominal interest rate (s < 1)**
\[ \begin{cases} 
\theta < \frac{\beta E(\theta)}{s_{\gamma}} & \ begin{align*} 
& h(\theta) = 0 \\
& a(\theta) > 0 \\
& l(m, a_{-1}, \theta) = py(\theta) + z(\theta)(1 - s_{\gamma}) + s_{\gamma} m - a_{-1} - T \\
& V_z = \tilde{V}_z(z) + \beta E(\theta)\left(\frac{1}{\gamma} - 1\right) 
\end{align*} \\
\theta \geq \frac{\beta E(\theta)}{s_{\gamma}} & \ begin{align*} 
& h(\theta) = 0 \\
& a(\theta) = 0 \\
& l(m, a_{-1}, \theta) = py(\theta) + z(\theta) - m - a_{-1} - T \\
& V_z = \tilde{V}_z(z) + \frac{\beta E(\theta)}{\gamma} 
\end{align*} 
\end{cases} \tag{28} \]

Lemma 4 and equation 5 shows the characteristics of the policy functions in two cases. When nominal interest rate is negative (27), and when it is positive (28). In an equilibrium with negative nominal interest rate households choose to hold all of their portfolio in terms of money. Money has a greater return comparing to bond in this case and households decide to hold all of their precautionary savings in terms of money. Higher amount of portfolio from previous period \((m, a_{-1})\) reduces labor supply \(l(m, a_{-1}, \theta)\), households choose to work less because of higher value of their asset portfolio.

Lemma 4 shows that the equilibrium is partially block recursive. Households do not need to know the distribution of the asset holding for their decision problems and prices \((p, s, w)\) contain all the information they need about the distributions in the aggregate economy. In the next section I show that we cannot have negative nominal interest rate in the stationary equilibrium and households’ policy functions can be described by 28.

3 Stationary Equilibrium

Here, I characterize the stationary equilibrium.

Definition 1. A stationary equilibrium is the set of households’ value functions \((W, B, V, \tilde{V})\); household choices \((y, l, z, a, h, q, b, L_1, L_2, \pi_1, \pi_2)\); firm choices \((Y, dN(q, b))\); prices \((p, s, w)\); which satisfy the following conditions:

1. Given the prices \((p, s, w)\), realization of shocks \(\theta\), asset balances and terms of trade in all submarkets \((q, x)\), household choices solve households’ optimality conditions (28 and 27)

2. Given prices and the terms of trade in all submarkets, firms maximize profit (1)

3. Free entry condition (3)

4. Stationarity
5. Symmetry

6. Bond market clears (29), labor market clears (30), and general goods market clears \((p = 1)\)

In the bond market the total amount of bonds supplied equals the sum of demanded bonds by households of different type. The nominal amount of supplied bond by the central bank is \(A_s\), and therefore the real supply of bond is \(\frac{A_s}{wM}\). The market clearing for bonds gives

\[
\frac{A_s}{wM} = \int \int \int \theta a dF(\theta)dG(m)dH(a-1) = \int \theta a(\theta) dF(\theta)
\]

(29)

where in the second equality, I use the fact that households’ decision on their asset holding is only based on their labor supply shock.

**Lemma 5.** No positive bond supply \((\lambda > 0)\) can support an equilibrium with negative nominal interest rate \((s \geq 1)\). Household’s choose to hold bonds as precautionary saving and they only choose money for transaction purposes:

- \(h(\theta) = 0\)
- \(z(\theta) \geq 0\)

From condition 27 and bond market clearing condition 29, its straightforward to show that positive amounts of bond supply would not clear the market when \(s \geq 1\). With positive real interest rate, households never use money for precautionary motives.

There are two cases for the equilibrium: First, when \(\bar{\theta} < \frac{\beta E(\theta)}{s\gamma} < \theta\), households with low enough \(\theta\) choose to hold positive amount of bond (traders in the asset market), while households with high \(\theta\) only hold money for transaction purposes(non-traders in the asset market). Figure 3 shows that the threshold \(\frac{\beta E(\theta)}{s\gamma}\) determines who participates in the asset market.

![Segmented asset market](Image)

Figure 2: Segmented asset market
Second, in the case where $\bar{\theta} < \frac{\beta E(\theta)}{s^{\gamma}}$, all of the households hold a portfolio of bond and money. In figure 3, $\frac{\beta E(\theta)}{s^{\gamma}}$ is very high and everybody participates in the market for bond. In deciding whether to participate in the asset market, households compare two scenarios: 1-working this period and buying bonds and redeeming purchased bonds for money next period, and 2-not working today and working tomorrow.

![Figure 3: Asset market with no segmentation](image)

From lemma 5 and equations 28, 17 and 5, I can show that the general shape for households’ money holding and bond holding is similar to figure 4. Conditional on participating (/not participating) in the asset market, changes in bond supply ($\lambda$) would only change the threshold ($\frac{\beta E(\theta)}{s^{\gamma}}$), and policy functions regarding the composition of households’ portfolio are not affected by open-market operations. Figure 4 shows that an equilibrium with segmented asset market arise when $\theta > \frac{\beta E(\theta)}{s^{\gamma}} > \bar{\theta}$. In an equilibrium with segmented asset market open-market operations affects the decision of the households regarding the composition of their real portfolio of assets for households at the participation margin and therefore has real effects on the distributions of asset portfolios in the economy. This property of the equilibrium is completely endogenous. In section 5 I will impose exogenously segmented asset market, and show that most of the results hold under this assumption.

Lemma 4 shows that when we have an equilibrium with no segmentation in the asset market, money holding from the previous period does not affect agents’ labor supply. In the same type of equilibrium households’ bond holding has a negative effect on their labor supply. This property of the equilibrium is due to the fact that households in an asset market with no segmentation and households with good shocks in a segmented asset market ($\theta < \frac{\beta E(\theta)}{s^{\gamma}}$) hold money balances only for transaction purposes. Note that this is different from the pure precautionary motive ($h(\theta) > 0$), and bond always dominates money because of positive real interest rate ($s < 1$). In a segmented asset market households with bad shocks ($\theta \geq \frac{\beta E(\theta)}{s^{\gamma}}$) consider unmatched buyers’ expected money balances as a precautionary motive for saving.

As shown in the appendix A, the labor market clearing condition is

---

16Note that with positive bond supply, we cannot have the case in which $\frac{\beta E(\theta)}{s^{\gamma}} < \theta$.  

19
Figure 4: Households’ asset portfolio for $\theta < \frac{\beta E(\theta)}{s\gamma} < \bar{\theta}$ and $\bar{\theta} < \frac{\beta E(\theta)}{s\gamma}$

$$\frac{1}{w\gamma} [\gamma - 1 + \lambda(s - 1)] = \int_{\theta}^{\bar{\theta}} a(\theta) dF(\theta) + (1 - \frac{1}{\gamma}) \int_{\theta}^{\bar{\theta}} \pi_1(z(\theta))(1 - b(L_1(z(\theta))))L_1(z(\theta))dF(\theta)$$

$$+ (1 - \frac{1}{\gamma}) \int_{\theta}^{\bar{\theta}} \pi_2(z(\theta))(1 - b(L_2(z(\theta))))L_2(z(\theta))dF(\theta) - \frac{1}{\gamma} \int_{\theta}^{\bar{\theta}} h(\theta)dF(\theta)$$

(30)

Using lemmas 4 and 5 and the fact that households do not save money for precautionary motive, I can summarize the market clearing conditions to a single equation that could be solved for bond price ($s$)

$$[\frac{\gamma - 1}{\lambda} + s - 1 + s\gamma^2 - 2\gamma] \int_{\theta}^{\bar{\theta}} a(\theta) dF(\theta) =$$

$$\gamma(1 - \gamma) \int_{\theta}^{\bar{\theta}} \pi_1(z(\theta))(1 - b(L_1(z(\theta))))L_1(z(\theta))dF(\theta)$$

$$+ \gamma(1 - \gamma) \int_{\theta}^{\bar{\theta}} \pi_2(z(\theta))(1 - b(L_2(z(\theta))))L_2(z(\theta))dF(\theta)$$

(31)
Note that equation 31 cannot solely be used for numerical computations, and we need to compute wage (30) and check for positive wages. From equations 28, 29 and 31, I can characterize the set of prices in equilibrium. Let $\bar{s}$ and $s$ as

$$\bar{s} = \frac{\beta E(\theta)}{\theta \gamma}$$

$$s = \frac{\beta E(\theta)}{\theta \gamma}$$

Let bond price be in the range: $\bar{s} \leq s$. The left hand side of 31 is 0 while the right hand side is a positive number. In this case there is no equilibrium. Previously, I have shown that $s < 1$ in equilibrium. Therefore, the market clears at a price in the range $s < \min\{1, \bar{s}\}$.

Let define $\zeta(\gamma, \lambda)$ as

$$\zeta(\gamma, \lambda) = \sum_{i=1,2} \int_{\theta}^{\bar{s}} \pi_i(z(\theta))(1 - b(L_i(z(\theta))))L_i(z(\theta))dF(\theta) \quad (32)$$

For the case where $s < \bar{s}$, the right graph on figure 4 shows the policy functions. In this case $\zeta(\gamma, \lambda)$ is independent of $\lambda$, and I show it by $\zeta_1(\gamma)$. The only policy variable on the right hand side of 31 is $\gamma$. For a constant positive rate of inflation ($\gamma > 1$) the left hand side shows a positive relationship between bond price and bond supply. From figure 4 and the positive relationship between bond price and bond supply, I can summarize the demand for bond in figure 5.

![Figure 5: Demand for bond ($\gamma > 1$)](image)

For low bond price, the return on bond is high enough to attract all of the households to
the asset market. They hold a positive portfolio of money and bond according to the right graph on figure 4. With higher bond price (lower interest rate) we have a segmented asset market and higher price in this type of equilibrium leads to low asset market participation.

From 31 and 32 theorem 2 follows

**Theorem 2.** For a positive rate of inflation ($\gamma > 1$) there exists the following thresholds for bond supply

$$
\lambda_l(\gamma) = \frac{1}{\gamma - 1} - \frac{\gamma \zeta_1(\gamma)}{\int_0^\theta a(\theta) dF(\theta)}
$$

$$
\lambda_u(\gamma) = \frac{1}{2\gamma + 1} - \frac{1}{\gamma - 1} - \frac{\gamma \zeta_1(\gamma)}{\int_0^\theta a(\theta) dF(\theta)}
$$

1. For $\lambda_l(\gamma) < \lambda < \lambda_u(\gamma)$ the policy functions are similar to the right graph in figure 4. Open-market operations have no effects on the distribution of real asset holdings. Open-market purchase (sale) of bond reduces (increases) bond yield and labor supply.

2. For $\lambda \geq \lambda_u(\gamma)$ the policy functions are similar to the left graph on figure 4. Open market operations affect the distribution of asset holding.

Theorem 2 shows an important property of the equilibrium. For high enough bond supply asset market is segmented and pure open-market operations has effects on the real distributions in the economy. In this case supplying more bonds will increase (decrease) the price (yield) of bonds. Fewer households decide to participate in the asset market due to lower return on bonds. They choose to hold more money for transaction purposes. This effect happens only when bond supply is high enough ($\lambda > \lambda_u(\gamma)$). Note that in the expression for $\lambda_u(\gamma)$ the exogenous variable $z(\theta)$ is independent of $\lambda$ for $\lambda < \lambda_u(\gamma)$. In order to compute $\lambda_u(\gamma)$ we can start with a low $\lambda$ and compute $\lambda_u(\gamma)$ and check if $\lambda < \lambda_u(\gamma)$.

Previously, I have shown that there is a positive relationship between bond price ($s$), and bond supply ($\lambda$) when bond price is in the range $[\lambda_l(\gamma), \lambda_u(\gamma)]$\textsuperscript{17}. Using this relationship I can prove the following lemma which shows that the positive relationship exists for higher bond supply ($\lambda > \lambda_u(\gamma)$).

**Lemma 6.** For $\lambda > \lambda_u(\gamma)$, bond price is increasing in bond supply ($\frac{\partial s}{\partial \lambda} > 0$).

The proof is in the appendix B. Using the above lemma I can find the threshold bond supply for equilibrium existence. Lets define $s_{\text{max}} = \min \{1, \pi\}$. $\lambda_{\text{max}}(\gamma)$ is the bond supply that solves the following equation

\textsuperscript{17}From equation 31
\[
\frac{\gamma - 1}{\lambda_{\text{max}}(\gamma)} + s_{\text{max}} - 1 + s_{\text{max}}\gamma^2 - 2\gamma \int_{\theta}^{\beta E(\theta)} a(\theta) dF(\theta) = \zeta(\gamma, \lambda_{\text{max}}(\gamma)) \quad (35)
\]

For a bond supply greater than \(\lambda_{\text{max}}(\gamma)\), the demand for bond falls to zero and bond market does not clear. Note that from equation 35, \(\lambda_{\text{max}}\) cannot be solved analytically. \(\zeta(\gamma, \lambda_{\text{max}}(\gamma))\) is a function of lottery choices of households and it is not clear how bond supply (\(\lambda\)) affects it. In section 4 I use numerical methods to calculate \(\lambda_{\text{max}}\) for different policy variables (\(\lambda, \gamma\)).

### 3.1 Welfare analysis

I have shown in the appendix C that the steady state welfare can be calculated using the following expression

\[
\varpi = \int \left[ U(y(\theta)) - \theta y(\theta) + u(q(z(\theta))) - \theta z(\theta) - \theta h(\theta) - s\gamma \theta a(\theta) \right] dF(\theta)
\]

\[
+ \left[ \int_{\theta}^{\pi_1(z(\theta))} \frac{L_1(z(\theta))}{\gamma} dF(\theta) \right] \int \theta dF(\theta)
\]

\[
+ \left[ \int_{\theta}^{\pi_2(z(\theta))} \frac{L_2(z(\theta))}{\gamma} dF(\theta) \right] \int \theta dF(\theta)
\]

\[
+ (1 + \frac{1}{\gamma}) \left[ \int a_{-1} dH_{a_{-1}} \right] \int \theta dF(\theta) + \frac{1}{\gamma} \left[ \int h_{-1} dJ_{h_{-1}} \right] \int \theta dF(\theta)
\]

\[
+ \frac{1}{w\gamma} [\gamma - 1 - \lambda + s\lambda] \int \theta dF(\theta)
\]

Using lemma 5 and equations 30 and 32 the measure of welfare can be simplified as

\[
\varpi = \int \left[ U(y(\theta)) - \theta y(\theta) \right] dF(\theta) + \int \left[ u(q(z(\theta))) - \theta z(\theta) - s\gamma \theta a(\theta) \right] dF(\theta)
\]

\[
+ \left[ \zeta(\gamma, \lambda) + (2 - s + 1 + \frac{1}{\gamma}) \int a(\theta) dF(\theta) \right] \int \theta dF(\theta) \quad (36)
\]

The following lemma shows the welfare effects of open-market operations

**Lemma 7.** In the case where \(s < \tilde{s} = \frac{\beta E(\theta)}{\theta}\gamma\), marginal increase/decrease in bond supply decrease/increase welfare.

With \(s < \tilde{s} = \frac{\beta E(\theta)}{\theta}\gamma\), from lemma 4 we can see that none of the policy functions are affected by marginal open-market operations (change in \(\lambda\)). The welfare measure can be
stated as
\[
\varpi = \int [U(y(\theta)) - \theta y(\theta)]dF(\theta) + \int [u(q(z(\theta))) - \theta z(\theta) - s\gamma \theta a(\theta)]dF(\theta)
+ \left[\zeta_1(\gamma) + (2 - s + 1 + \frac{1}{\gamma}) \int a(\theta)dF(\theta)\right] \int \theta dF(\theta)
\]  
(37)

Therefore, it is straightforward to see that in the above expression changes in \( \lambda \) only affects bond price \((s)\). The above measure of welfare is decreasing in \( s \). In the above case labor market clearing is

\[
\frac{1}{w\gamma}[\gamma - 1 - \lambda + s\lambda] = (2 - s\gamma) \int_\theta^\eta a(\theta)dF(\theta) + (1 - \frac{1}{\gamma})\zeta_1(\gamma)
\]  
(38)

The above equation for wage shows that a marginal increase in \( \lambda \) increases wage \((w)\). I can state lemma 7 as follows

**Corollary 3.** In an equilibrium with no market segmentation, marginal open-market purchase of bond increases welfare.

Using this framework, analytical investigation of the welfare effects of policies in a segmented asset market is not possible. In the next section I will use numerical methods to calculate welfare under different policies.

### 4 Numerical Example

In order to simulate the economy, I use the following algorithm:

1. For given supply of bonds \((\lambda)\) and inflation \((\gamma)\), and an arbitrary bond price \((s)\) calculate policy functions \((a(\theta), b(\theta), z(\theta), y(\theta), l(\theta), b(z(\theta)), q(z(\theta)))\) (4) and lottery choices \((\pi_1(z(\theta)), \pi_2(z(\theta)), L_1(z(\theta)), L_2(z(\theta)))\) (17)

2. Calculate the value functions \((B(z(\theta)), \tilde{V}(z(\theta)))\)

3. Calculate wage \((w)\) using labor market clearing condition 30

4. If \(w < 0\) change \(s\) and start from 1.

5. Check bond market clearing condition 29, adjust bond price and start from 1. until bond market clears.
I simulate the economy using the following functional forms:

\[
\begin{align*}
    u(c) &= u_0 \frac{(c + a)^{1-\sigma} - a^{1-\sigma}}{1-\sigma};
    U(c) &= U_0 \frac{(c + a)^{1-\sigma_u} - a^{1-\sigma_u}}{1-\sigma_u} \\
    \psi(q) &= \psi_0 q^\phi; 
    \mu(b) = 1 - b; 
    F(\theta) \text{ is continuous uniform on } [\underline{\theta}, \overline{\theta}]
\end{align*}
\]

I use the following parameter values:


Figure 6 shows different types of equilibria for different amounts of bond supply and inflation rate. As theorem 2 shows for low amount of bond supply there is no equilibrium (\( \lambda < \lambda_l(\gamma) \)). For medium amount of bond supply the bond price is low and the equilibrium shows no segmentation in the asset market (\( \lambda_l(\gamma) < \lambda < \lambda_u(\gamma) \)). High real interest rate attracts all of the households to the asset market and they hold positive portfolio composed of bond and money. Under a higher bond supply (\( \lambda_u(\gamma) < \lambda < \lambda_{\max} \)) a segmented asset market arises. Return on bond attracts household with high income while low income households choose to hold money. Under a policy of high bond supply (\( \lambda > \lambda_{\max} \)) no equilibrium exists.

Figure 6: Criteria for different types of equilibria

Figure 7 shows the policy functions regarding households’ portfolio and the effects of open-market operations. In a segmented asset market, high income households choose a
positive portfolio of bond and money. The threshold that determines who participates in the asset market is affected by open-market operations. As it can be seen in figure 7 an open-market purchase of bond would increase the real interest rate and shift the threshold to the right. More households decide to participate in the bond market as a result of an open-market purchase of assets.

Figure 7: Choice of asset holding in a segmented asset market

Figure 8 shows the asset portfolio choice of households in an equilibrium with no segmentation. Comparing to figure 7, here bond supply is so low that high real interest rates attract all of the households to the asset market and they hold a positive portfolio composing of money and bond. A marginal policy of pure open-market operation would not affect the decision of households regarding their real asset holding, and would not have real effects on the distributions in the economy.

Figure 9 shows labor supply in an equilibrium with no segmentation in the asset market. Households with bad shocks (high $\theta$) and high asset balance work less. Households with good shocks (low $\theta$) work more and hold more money for transaction purposes. A pure policy of open-market operations (marginal change in $\lambda$) would shift the labor supply. Higher real interest rate would change labor supply of households but households’ real asset holding would not change (Figure 8).

Figure 10 shows labor supply in an equilibrium with segmented asset market. Households with bad shocks ($\theta \geq \frac{\beta E(\theta)}{\gamma}$) supply labor only to fund their money holding for the next subperiod ($z(\theta)$). Households who received better shocks than the threshold for asset market
Figure 8: Choice of asset holding in an equilibrium with no segmentation

Figure 9: Labor supply in an equilibrium with no segmentation ($\gamma = 1.01, \lambda = 0.003$)
participation ($\theta < \frac{\beta E(\theta)}{s\gamma}$) provide high labor supply to buy bonds ($a(\theta)$) as a precautionary saving for the next period. A pure policy of open-market operations (marginal changes in $\lambda$) has two effects: First, it has a level effect on the labor supply. This is similar to the case with no segmentation. Higher real interest rates requires higher labor supply for the same real asset holding. Second, open-market operations changes the threshold ($\frac{\beta E(\theta)}{s\gamma}$), and therefore affects the participation decision of households in the market for bonds. Higher real interest rate, attracts some of households who were not participating in the asset market, and these households supply more labor.

![Figure 10: Labor supply in an equilibrium with segmented asset market ($\gamma = 1.01, \lambda = 0.0085$)](image)

Figure 10 shows the characteristics of submarkets in the decentralized market. Agents with higher money holdings search in submarkets with higher price, output and matching probabilities. This property of the equilibrium is shared with many competitive search models\(^\text{18}\). Agents sort themselves according to their money holdings. Households with higher money balances have low marginal value for money. As shown in lemma 2, they decide to get rid of a high amount of money as soon as they can and choose submarket with higher price and higher matching probability compared to households with low money balances. Unlike models of bargaining, buyers and sellers know the marginal value of money holdings of all of the households in the economy and they commit to posted terms of trade.

\(^{18}\text{e.g. equilibrium in Menzio et al. (2011) and Sun (2012) shows similar properties.}\)
Figures 11 and 12 show the output choice of households in the decentralized market. Generally, households with better shocks participate in submarkets with higher output. As shown in figures 7 and 8, conditional on participation (not participation) in the asset market, households with better shocks choose higher amounts of money balances. Figure 11 shows that households with higher money balances choose higher output. Therefore, we can see that conditional on participating (not participating) in the asset market households with better shocks choose submarkets with higher output and figure 12 confirms this. In the case with a segmented asset market (figure on the right) a marginal open-market purchase of bond would increase asset market participation and reduce real output choice of households on the participation margin to lower values.

Figure 13 shows the matching probability choice of households in the decentralized market. As shown in figures 7 and 8, conditional on participation (not participation) in the asset market, households with better shocks choose higher amounts of money balances. Figure 11 shows that households with higher money balances choose submarkets with higher matching probabilities. As a result, conditional on participating (not participating) in the asset market households with better shocks choose submarkets with higher matching probability and figure 13 confirms this. In the case with a segmented asset market (figure on the right) a marginal open-market purchase of bond would increase asset market participation and reduce real output choice of households on the participation margin to lower values.

I can discuss the effects of open-market operations on the extensive and intensive margins using figures 12 and 13. A marginal open-market purchase of bond would decrease $\lambda$ and bond price $(s)$. This policy will have no effects on the intensive margin (right graph in 12) and extensive margin (right graph in 13) when we have an asset market with no
Figure 12: Output choice of households in decentralized market

Figure 13: Matching probability choice of households in decentralized market
segmentation. As shown in figures 12 and 13 In a segmented asset market, open market purchase of bonds would shift the threshold for asset market participation to the right. This will decrease both the intensive and the extensive margins of trade for households in the participation margin. Higher real interest rate attracts a subset of households to the bond market. In the decentralized market these households choose to apply to submarkets with lower matching probability and lower output and this will decrease both the extensive margin and the intensive margin of trade.

Figure 14 shows that conditional on participating (not participating) in the asset market, household with better income shock pay lower price per unit of output in the decentralized market.

Figure 14: Price per unit choice of households in decentralized market

Figures 15 shows welfare for different values of bond supply and inflation rate. The central bank can generally improve welfare by purchasing bonds and supplying money. The policy of open-market purchase of bond is most effective when the asset market is segmented. This policy would increase the participation rate in the asset market and help households smooth consumption. By participating in the asset market, households are able to better insure themselves against idiosyncratic income shocks. When the asset market is not segmented, marginal open-market purchase/sale of bond would only change the real interest rate, and has limited welfare effects.

Figure 16 shows equilibrium bond price ($s$) for different amounts of bond supply ($\lambda$) and different inflation rates ($\gamma$). At each level of inflation bond price increases with higher supply of bonds\textsuperscript{19}.

\textsuperscript{19}Note that the price of bond is the inverse of nominal return on bonds
Figure 15: Welfare

Figure 16: Bond prices
Figure 17 shows equilibrium wage ($w$) for different amounts of bond supply ($\lambda$). For a fixed rate of inflation, wage increases with bond supply.

Figure 17: Wage

5 Exogenously segmented asset market

In this section I introduce another source of heterogeneity to the model. Following Alvarez et al. (2001), I assume only a fixed fraction of households attend the asset markets (traders), and the remaining never has access to the asset market (non-traders). This extension allows me to compare the results of this paper to the literature that assumes the asset markets are exogenously segmented\textsuperscript{20}. Theorem 4 shows that the same logic from the case with endogenous asset market segmentation applies and asset market traders and non-traders solve optimization problems similar to the problem in the previous sections. The households’ decisions are only linked through the market clearing conditions and prices. Households do not take into account the distribution of asset holdings among traders and non-traders. The following theorem shows that the main results in the previous sections are robust to adding exogenously segmented asset market.

Theorem 4. With exogenously segmented asset markets, value functions, policy functions and labor choices of traders in the asset market have the same properties as the case without

\textsuperscript{20}e.g. Alvarez et al. (2001), Khan and Thomas (2010) and Chiu (2007)
exogenously segmented asset market.

The formal proof is in the appendix D.

6 Concluding Remarks and Possible Extensions

This paper studies the central bank’s open-market operations in a model with heterogeneous agents. Using competitive search in the frictional market for goods allows me to study the distribution of asset holding in a tractable model. The central bank can implement monetary policy by supplying money and trading bond in the asset market. There are two types of equilibria. In an equilibrium with low bond supply, the asset market is not segmented. All of the agents attend the asset market and hold positive portfolio of bond and money. In an equilibrium with high bond supply, segmentation is generated endogenously. Households with good income shocks attend the asset market and hold positive portfolio of bond and money. Household with low income only hold money in their portfolio.

In an equilibrium with no segmentation, open-market operations have no real effects on the distribution of real asset holding. In an equilibrium with segmented asset market, open-market operations change the decision of a subset of households and have real effects on the distribution of asset holdings. The main results are robust to exogenously segmented asset market.

One possible extension of the model is to relax the quasi-linear preference of the households to a more general preference structure. With a more general preference structure, one can study the wealth effects in this framework. In this setup, it would be difficult to analytically show some of the properties of the equilibrium and computational exercise would be more critical. By adding aggregate shocks to the economy, one can do an analysis similar to Krusell and Smith (1998) with this framework.
Appendix

A Market clearing conditions

I can find the cumulative distribution of money before lotteries by:

\[ G(m) = \int \int \theta dF(\theta) dH \]  (39)

and similarly the distribution of bond before lotteries follows:

\[ H(a_{-1}) = \int \int \theta dF(\theta) dG \]  (40)

I assume a balanced budget for government at each period of time. The total real transfer that a household receives is the sum of transfers from printing money and the transfers received from bond market:

\[ T = \frac{\gamma - 1}{w\gamma} + \frac{sA - A_{-1}}{wM'} \]  (41)

In the bond market the total amount of bonds supplied equals the sum of demanded bonds by households of different type. Thus, the market clearing for bonds gives:

\[ \frac{As}{wM} = \int \int \theta dF(\theta) dG(m) dH(a_{-1}) \]  (42)

In the general-good market, the market clearing condition is:

\[ Y = \int \theta y(\theta) dF(\theta) \]  (43)

LD is the same as Sun (2012):

\[ LD = Y + \int_{\theta} \frac{\pi_1(z(\theta))b(L_1(z(\theta)))}{\mu(b(L_1(z(\theta))))} \left[ k + \psi(q(L_1(z(\theta)))) \mu(b(L_1(z(\theta)))) \right] dF(\theta) \]

\[ + \int_{\theta} \frac{\pi_2(z(\theta))b(L_2(z(\theta)))}{\mu(b(L_2(z(\theta))))} \left[ k + \psi(q(L_2(z(\theta)))) \mu(b(L_2(z(\theta)))) \right] dF(\theta) \]  (44)

The firms zero-profit condition gives:
Then LD becomes:

\[
LD = \int_{\hat{\theta}}^{\theta} y(\theta)dF(\theta) + \int_{\hat{\theta}}^{\theta} \pi_1(z(\theta))b(L_1(z(\theta))) \mu(L_1(z(\theta)))L_1(z(\theta))dF(\theta)
+ \int_{\hat{\theta}}^{\theta} \pi_2(z(\theta))b(L_2(z(\theta)))L_2(z(\theta))dF(\theta)

\]  

(45)

Aggregate labor supply is the sum of households labor supply:

\[
LS = \int_{\hat{\theta}}^{\theta} \int \int [y(\theta) + z(\theta) + s\gamma a(\theta) - m - a_{-1} - T]dF(\theta)dG_a(m)dH(a_{-1})
\]

in which \(dG_m\) is the distribution of money holdings at the beginning of the period. Substituting for \(l\) in the above equation we get

\[
LS = \int_{\hat{\theta}}^{\theta} \int \int [py(\theta) + z(\theta) + s\gamma a(\theta) - m - a_{-1} - T]dF(\theta)dG_a(m)dH(a_{-1})
\]

(46)

Substituting for \(T\):

\[
LS = \int_{\hat{\theta}}^{\theta} \int \int [py(\theta) + z(\theta) + s\gamma a(\theta) - m - a_{-1} - \frac{\gamma - 1}{w\gamma - sA}mG_a(m)dH(a_{-1})
\]

\[
- \frac{sA}{wM'} \int \int [y(\theta) + z(\theta) + s\gamma a(\theta)]dF(\theta)dG_a(m)dH(a_{-1})
\]

(47)

LS becomes:

\[
LS = \frac{A_{-1}}{w\gamma M} - \frac{sA}{wM'} - \frac{\gamma - 1}{w\gamma} - \int mdG_a(m) - \int a_{-1}dH(a_{-1})

\]

\[
+ \int_{\hat{\theta}}^{\theta} [y(\theta) + z(\theta) + s\gamma a(\theta)]dF(\theta)
\]

(48)

Labor market clearing condition gives:
\[
\int_{\tilde{\theta}} s\gamma a(\theta)dF(\theta) + \int_{\tilde{\theta}} \pi_1(\theta)(1 - b(L_1(\theta)))L_1(\theta)dF(\theta) \\
+ \int_{\tilde{\theta}} \pi_2(\theta)(1 - b(L_2(\theta)))L_2(\theta)dF(\theta) \\
= \frac{sA}{wM} + \frac{\gamma - 1}{w\gamma} - \frac{A_{-1}}{w\gamma M} + \int mdG_a(m) + \int a_{-1}dH(a_{-1})
\]

\(m\) is the distribution of money at the beginning of the period. Therefore, it consists of balances that are not spent plus the payments on nominal bonds:

\[
\int mdG_a(m) = \\
\int_{\tilde{\theta}} \pi_1(\theta)\left[1 - b(L_1(\theta))\right] \frac{L_1(\theta)}{\gamma} dF(\theta) \\
+ \int_{\tilde{\theta}} \pi_2(\theta)\left[1 - b(L_2(\theta))\right] \frac{L_2(\theta)}{\gamma} dF(\theta) + \int \frac{a_{-1}}{\gamma} dH(a_{-1}) + \int \frac{h_{-1}}{\gamma} dJ_{h_{-1}}
\]

Plug in the labor market clearing condition:

\[
\frac{sA}{wM} + \frac{\gamma - 1}{w\gamma} - \frac{A_{-1}}{w\gamma M} = \\
\int_{\tilde{\theta}} s\gamma a(\theta)dF(\theta) + (1 - \frac{1}{\gamma}) \int_{\tilde{\theta}} \pi_1(\theta)\left[1 - b(L_1(\theta))\right]L_1(\theta)dF(\theta) \\
+ (1 - \frac{1}{\gamma}) \int_{\tilde{\theta}} \pi_2(\theta)\left[1 - b(L_2(\theta))\right]L_2(\theta)dF(\theta) - \int \frac{a_{-1}}{\gamma} dH(a_{-1}) - \int \frac{h_{-1}}{\gamma} dJ_{h_{-1}}
\]

The labor-market-clearing can be written as:

\[
\frac{1}{w\gamma} [\gamma - 1 - \lambda + s\lambda] = \\
(2 - s\gamma) \int_{\tilde{\theta}} a(\theta)dF(\theta) + (1 - \frac{1}{\gamma}) \int_{\tilde{\theta}} \pi_1(\theta)\left[1 - b(L_1(\theta))\right]L_1(\theta)dF(\theta) \\
+ (1 - \frac{1}{\gamma}) \int_{\tilde{\theta}} \pi_2(\theta)\left[1 - b(L_2(\theta))\right]L_2(\theta)dF(\theta) - \int \frac{h(\theta)}{\gamma} dF(\theta)
\]
B  Proof of Lemma 6

Proof by contradiction: Let’s assume that bond price \((s)\) is not increasing in bond supply \((\lambda)\). There exist \(\lambda_1\) and \(\lambda_1 + \epsilon\) such that \(s_{\lambda_1} = s_{\lambda_1 + \epsilon}\) and \(\epsilon \neq 0\). According to lemma 4 and figure 4, we have the same threshold for asset market participation

\[
\frac{\beta E(\theta)}{s_{\lambda_1}} = \frac{\beta E(\theta)}{s_{\lambda_1 + \epsilon}}
\]

The policy functions of the households in these two cases are the same and therefore we have the same distribution of asset holdings among the agents. The bond market clearing condition equates the bond supplys which is a contradiction since \(\epsilon \neq 0\).

C  Welfare Analysis

I use the household’s utility function to calculate welfare:

\[
\varpi = \int \int \int \{U(y) + u(q) - \theta l\}dF(\theta)dG(m)dH(a_{-1})
\]

\[
= \int U(y(\theta))dF(\theta) + \int u(q(z(\theta)))dF(\theta) - \int \int \int \{\theta l\}dF(\theta)dG(m)dH(a_{-1})
\]

I can write the last integral as:

\[
\int \int \int (\theta l)dF(\theta)dG(m)dH(a_{-1}) =
\]

\[
\int [\theta(y(\theta) + z(\theta) + h(\theta) + s_\gamma a(\theta))]dF(\theta)
\]

\[
- \int \theta \left( \int mdG_a \right) dF(\theta) - \int \theta \left( \int a_{-1}dH(a_{-1}) \right) dF(\theta) - T \int \theta dF(\theta)
\]

I have shown in the market clearing appendix the distribution of money before the lotteries is:

\[
\int mdG_a(m) =
\]

\[
\int \pi_1(z(\theta))[1 - b(L_1(z(\theta)))]\frac{L_1(z(\theta))}{\gamma}dF(\theta)
\]

\[
+ \int \pi_2(z(\theta))[1 - b(L_2(z(\theta)))]\frac{L_2(z(\theta))}{\gamma}dF(\theta) + \int \frac{a_{-1}}{\gamma}dH(a_{-1}) + \int \frac{h_{-1}}{\gamma}dJ_{h_{-1}}
\]
I can substitute for the distribution of money \((md\gamma)\) and labor supply \((l)\) from the above equations, and for government transfers \((T)\) from the the market clearing appendix to simplify the equation for welfare:

\[
\varpi = \int [U(y(\theta)) - \theta y(\theta) + u(q(z(\theta))) - \theta z(\theta) - \theta h(\theta) - s\gamma \theta a(\theta)]dF(\theta)
\]

\[
+ \left\lfloor \int_{\theta}^{\bar{\theta}} \pi_1(z(\theta))\left[1 - b(L_1(z(\theta)))\right] \frac{L_1(z(\theta))}{\gamma} dF(\theta) \right\rfloor \int \theta dF(\theta)
\]

\[
+ \left\lfloor \int_{\theta}^{\bar{\theta}} \pi_2(z(\theta))\left[1 - b(L_2(z(\theta)))\right] \frac{L_2(z(\theta))}{\gamma} dF(\theta) \right\rfloor \int \theta dF(\theta)
\]

\[
+ (1 + \frac{1}{\gamma}) \left\lfloor \int a_{-1} dH_{a_{-1}} \right\rfloor \int \theta dF(\theta) + \frac{1}{\gamma} \left\lfloor \int h_{-1} dJ_{h_{-1}} \right\rfloor \int \theta dF(\theta)
\]

\[
+ \frac{1}{w\gamma} [\gamma - 1 - \lambda + s\lambda] \int \theta dF(\theta)
\]

D Proof of theorem 4

Lets assume there are two types of agents in the economy, traders in the asset market (denoted by subscript T) and non-traders (denoted by subscript N).

Value function of a trader:

\[
W_T(m_T, a_{-1}, \theta) = \max_{y_T, l_T, z_T, a_T} U(y_T) - \theta l_T + V_T(z_T, h_T, a_T)
\]

\[st. \quad py_T + z_T + s\gamma a \leq m_T + a_{-1} + l_T + T\]

Value function of a non-trader:

\[
W_N(m_N, \theta) = \max_{y_N, l_N, z_N} U(y_N) - \theta l_N + V_N(z_N, h_N)
\]

\[st. \quad py_N + z_N \leq m_N + l_N + T\]

Using the budget constraint to eliminate \(l_{i=N,T}\):

\[
W_T(m_T, a_{-1}, \theta) = \theta(m_T + T + a_{-1}) + \max_{y_T \geq 0} \{U(y_T) - \theta py_T\} + \max_{z_T, a_T, h_T} \{-\theta(z_T + s\gamma a + h_T) + V_T(z_T, h_T, a_T)\}
\]
\[ W_N(m_N, \theta) = \theta(m_N + T) + \max_{y_N \geq 0} \{U(y_N) - \theta y_N\} + \max_{z_N, h_N} \{-\theta(z_N + h_N) + V_N(z_N, h_N)\} \]

The optimal choices of \( y_{i \in \{T, N\}} \), \( z_{i \in \{T, N\}} \) and \( a \) must satisfy:

\[ U'(y_T) = U'(y_N) = \theta \quad (50) \]

The above expression shows that a trader and a non-trader choose the same amount of consumption in the frictionless market:

\[ y_T(\theta) = y_N(\theta) = y(\theta) \]

\[
\begin{align*}
\frac{\partial V_T(z_T, h_T, a)}{\partial z_T} & \begin{cases} 
\leq \theta & z_T \geq 0 \\
\geq \theta & z_T \leq \overline{m} - s\gamma a - h_T
\end{cases} \\
\frac{\partial V_T(z_T, h_T, a)}{\partial h_T} & \begin{cases} 
\leq \theta & h_T \geq 0 \\
\geq \theta & h_T \leq \overline{m} - s\gamma a - z_T
\end{cases} \\
\frac{\partial V_T(z_T, h_T, a)}{\partial a} & \begin{cases} 
\leq \theta s\gamma & a \geq 0 \\
\geq \theta s\gamma & sa \leq \overline{m} - z_T - h_T
\end{cases}
\end{align*} \quad (51, 52, 53) \]

\[
\begin{align*}
\frac{\partial V_N(z_N, h_N)}{\partial z_N} & \begin{cases} 
\leq \theta & z_N \geq 0 \\
\geq \theta & z_N \leq \overline{m} - h_N
\end{cases} \\
\frac{\partial V_N(z_N, h_N)}{\partial h_N} & \begin{cases} 
\leq \theta & h_N \geq 0 \\
\geq \theta & h_N \leq \overline{m} - z_N
\end{cases}
\end{align*} \quad (54, 55) \]

The value functions can be written as:

\[ W_T(m_T, a_{-1}, \theta) = W_T(0, 0, \theta) + \theta m_T + \theta a_{-1} \quad (56) \]

Where:

\[ W_T(0, 0, \theta) = U(y(\theta)) - \theta y(\theta) + V_T(z_T(\theta), h_T(\theta), a(\theta)) - \theta(z_T(\theta) + h_T(\theta) + s\gamma a(\theta)) \quad (57) \]

\[ W_N(m_T, \theta) = W_T(0, \theta) + \theta m_N \quad (58) \]
Where:

\[
W_N(0, \theta) = U(y(\theta)) - \theta y(\theta) + V_N(z_N(\theta), h_N(\theta)) - \theta(z_N(\theta) + h_N(\theta))
\] (59)

We can see that the value function \(W()\) is linear in household’s asset holdings for both traders and non-traders. Agents problem in the frictional market for traders and non-traders are similar. The difference comes from their value function which has 3 state variables for traders and 2 state variables for non-traders. After simplification and applying the lotteries as the previous section I can write agents value function as:

\[
V_T(z_T, h_T, a) = \widetilde{V}_T(z) + \beta E\left[W_T\left(\frac{z_T + h_T}{\gamma}, a, \theta\right)\right]
= \widetilde{V}_T(z_T) + \beta E[W_T(0, 0, \theta)] + \frac{\beta E(\theta)z_T}{\gamma} + \frac{\beta E(\theta)h_T}{\gamma} + \beta E(\theta)a
\] (60)

\[
V_N(z_N, h_N) = \widetilde{V}_N(z) + \beta E\left[W_N\left(\frac{z_N + h_N}{\gamma}, \theta\right)\right]
= \widetilde{V}_N(z_N) + \beta E[W_N(0, \theta)] + \frac{\beta E(\theta)z_N}{\gamma} + \frac{\beta E(\theta)h_N}{\gamma}
\] (61)

Trader’s and non-trader’s choice of bond holding follows the following condition with complementary slackness:

\[
\begin{align*}
\{ & a(\theta) \geq 0 \quad \theta \geq \frac{\beta E(\theta)}{s^\gamma} \\
& a(\theta) \leq m_T - z_T(\theta) - h_T(\theta) \quad \theta \leq \frac{\beta E(\theta)}{s^\gamma} \\
& h_T(\theta) \geq 0 \quad \theta \geq \frac{\beta E(\theta)}{s^\gamma} \\
& h_T(\theta) \leq m_T - z_T(\theta) - a'(\theta) \quad \theta \leq \frac{\beta E(\theta)}{s^\gamma} \\
& h_N(\theta) \geq 0 \quad \theta \geq \frac{\beta E(\theta)}{s^\gamma} \\
& h_N(\theta) \leq m_N - z_N(\theta) \quad \theta \leq \frac{\beta E(\theta)}{s^\gamma}
\end{align*}
\] (62)

\[
\begin{align*}
& h_T(\theta) \geq 0 \quad \theta \geq \frac{\beta E(\theta)}{s^\gamma} \\
& h_T(\theta) \leq m_T - z_T(\theta) - a'(\theta) \quad \theta \leq \frac{\beta E(\theta)}{s^\gamma}
\end{align*}
\] (63)

\[
\begin{align*}
& h_N(\theta) \geq 0 \quad \theta \geq \frac{\beta E(\theta)}{s^\gamma} \\
& h_N(\theta) \leq m_N - z_N(\theta) \quad \theta \leq \frac{\beta E(\theta)}{s^\gamma}
\end{align*}
\] (64)

The labor choices of traders are the same as the labor choices in equations 28 and 27. Labor choices of non-traders are as 65:

\[
l_N(m, \theta) = \begin{cases} 
py(\theta) + z_N(\theta) - m_N - T_N & \theta > \frac{\beta E(\theta)}{s^\gamma} \\
py(\theta) + z_N(\theta) - m_N - T_N & \theta = \frac{\beta E(\theta)}{s^\gamma} \\
py(\theta) + m - m_N - T_N & \theta < \frac{\beta E(\theta)}{s^\gamma}
\end{cases}
\] (65)
D.1 Market clearing condition and welfare measure

Similar to the case where all of the agents trade in the asset market the real transfer is:

\[ T = \frac{\gamma - 1}{w\gamma} + \frac{sA - A_{-1}}{wM'} \]  

(66)

The market clearing condition for the bond market and the general good market is the same as 29 and 43.

Similar to the case with only one type of agent, the labor demand can be written as:

\[
LD = \int_0^\theta \int_0^\theta y(\theta)dF(\theta) + \int_0^\theta \pi_1(z_T(\theta))b(L_1(z_T(\theta)))L_1(z_T(\theta))dF_T(\theta) \\
+ \int_0^\theta \pi_N(z_N(\theta))b(L_1(z_N(\theta)))L_1(z_N(\theta))dF_N(\theta) \\
+ \int_0^\theta \pi_2(z_T(\theta))b(L_2(z_T(\theta)))L_2(z_T(\theta))dF_T(\theta) \\
+ \int_0^\theta \pi_2(z_N(\theta))b(L_2(z_N(\theta)))L_2(z_N(\theta))dF_N(\theta)
\]  

(67)

Labor supply is the sum of households labor supply:

\[
LS = \int_\theta^\theta \int_\theta^\theta l_T(m_T, a, \theta)dF_T(\theta)dG_a(m_T)dH(a_{-1}) + \int_\theta^\theta \int_\theta^\theta l_N(m_N, \theta)dF_N(\theta)dG_a(m_N)
\]

Substituting for labor choices and transfers:

\[
L_s = \int_\theta^\theta \int_\theta^\theta \int_\theta^\theta [py(\theta) + z_T(\theta) + s\gamma a(\theta) - m_T - a_{-1}]dF_T(\theta)dG_a(m_T)dH(a_{-1}) \\
+ \int_\theta^\theta \int_\theta^\theta \int_\theta^\theta [py(\theta) + z_N(\theta) - m_N]dF_N(\theta)dG_a(m_N) \\
- \frac{\gamma - 1}{w\gamma} - \frac{sA}{wM'} + \frac{A_{-1}}{w\gamma M}
\]  

(68)

\[
LS = \frac{A_{-1}}{w\gamma M} - \frac{sA}{wM'} - \frac{\gamma - 1}{w\gamma} - \int mTdG_a(m_T) - \int m_NdG_a(m_N) - \int a_{-1}dH(a_{-1}) \\
+ \int_\theta^\theta [y(\theta) + z_T(\theta) + s\gamma a(\theta)]dF_T(\theta) + \int_\theta^\theta [y(\theta) + z_N(\theta)]dF_N(\theta)
\]  

(69)
Labor market clearing condition gives:

\[\frac{A_{-1}}{w^\gamma M} - \frac{sA}{wM^t} - \frac{\gamma - 1}{w^\gamma} - \int m_T dG_a(m_T) - \int m_N dG_a(m_N)
- \int a_{-1} dH(a_{-1}) + \int [z_T(\theta) + sa(\theta)] dF_T(\theta) + \int [z_N(\theta)] dF_N(\theta)
= \int \pi_1(z_T(\theta)) b(L_1(z_T(\theta))) L_1(z_T(\theta)) dF_T(\theta) + \int \pi_1(z_N(\theta)) b(L_1(z_N(\theta))) L_1(z_N(\theta)) dF_N(\theta)
+ \int \pi_2(z_T(\theta)) b(L_2(z_T(\theta))) L_2(z_T(\theta)) dF_T(\theta) + \int \pi_2(z_N(\theta)) b(L_2(z_N(\theta))) L_2(z_N(\theta)) dF_N(\theta)\]

Similar to the appendix A:

\[\int m_T dG_a(m_T) =
\int \pi_1(z_T(\theta))[1 - b(L_1(z_T(\theta)))] \frac{L_1(z_T(\theta))}{\gamma} dF_T(\theta)
+ \int \pi_2(z_T(\theta))[1 - b(L_2(z_T(\theta)))] \frac{L_2(z_T(\theta))}{\gamma} dF_T(\theta) + \int \frac{a_{-1}}{\gamma} dH(a_{-1}) + \int \frac{h_{-1T}}{\gamma} dJ_{h_{-1T}}\]

where \(dG_a(m_T)\) is the traders’ distribution of money holdings at the beginning of the period. Similarly, I can state the same for distribution of money holding among non-traders

\[\int m_N dG_a(m_N) =
\int \pi_1(z_N(\theta))[1 - b(L_1(z_N(\theta)))] \frac{L_1(z_N(\theta))}{\gamma} dF_N(\theta)
+ \int \pi_2(z_N(\theta))[1 - b(L_2(z_N(\theta)))] \frac{L_2(z_N(\theta))}{\gamma} dF_N(\theta) + \int \frac{h_{-1N}}{\gamma} dJ_{h_{-1N}}\]

Plug in the labor market clearing condition:
\[
\frac{1}{w\gamma}[\gamma - 1 - \lambda + s\lambda] = \\
(2 - s\gamma) \int \bar{\theta} a(\theta)dF_T(\theta) + \left(1 - \frac{1}{\gamma}\right) \int \bar{\theta} \pi_1(z_T(\theta))(1 - b(L_1(z_T(\theta))))L_1(z_T(\theta))dF_T(\theta) \\
+ \left(1 - \frac{1}{\gamma}\right) \int \bar{\theta} \pi_2(z_T(\theta))(1 - b(L_2(z_T(\theta))))L_2(z_T(\theta))dF_T(\theta) - \int \frac{h_T(\theta)}{\gamma}dF_T(\theta) \\
+ \left(1 - \frac{1}{\gamma}\right) \int \bar{\theta} \pi_1(z_N(\theta))(1 - b(L_1(z_N(\theta))))L_1(z_N(\theta))dF_N(\theta) - \int \frac{h_N(\theta)}{\gamma}dF_N(\theta) \\
+ \left(1 - \frac{1}{\gamma}\right) \int \bar{\theta} \pi_2(z_N(\theta))(1 - b(L_2(z_N(\theta))))L_2(z_N(\theta))dF_N(\theta)
\]

It can be shown as in appendix for the benchmark model that the measure of welfare is:

\[
\varpi = \int [U(y(\theta)) - \theta y(\theta) - u(q(z_T(\theta))) - \theta z_T(\theta) - s\gamma\theta a(\theta)]dF_T(\theta) \\
+ \left[\int \bar{\theta} \pi_1(z_T(\theta))(1 - b(L_1(z_T(\theta))))L_1(z_T(\theta))dF_T(\theta)\right] \int \theta dF_T(\theta) \\
+ \left[\int \bar{\theta} \pi_2(z_T(\theta))(1 - b(L_2(z_T(\theta))))L_2(z_T(\theta))dF_T(\theta)\right] \int \theta dF_T(\theta) \\
+ \left(1 + \frac{1}{\gamma}\right) \left[\int a_{-1}dH_{a_{-1}}\right] \int \theta dF_T(\theta) + \frac{1}{\gamma} \left[\int h_{-1T}dJ_{h_{-1T}}\right] \int \theta dF_T(\theta) \\
- \frac{1}{w\gamma} [\gamma - 1 - \lambda + s\lambda] \int \theta dF_T(\theta) \\
+ \int [U(y(\theta)) - \theta y(\theta) + u(q(z_N(\theta))) - \theta z_N(\theta)]dF_N(\theta) \\
+ \left[\int \bar{\theta} \pi_1(z_N(\theta))(1 - b(L_1(z_N(\theta))))L_1(z_N(\theta))dF_N(\theta)\right] \int \theta dF_N(\theta) \\
+ \left[\int \bar{\theta} \pi_2(z_N(\theta))(1 - b(L_2(z_N(\theta))))L_2(z_N(\theta))dF_N(\theta)\right] \int \theta dF_N(\theta) \\
+ \frac{1}{\gamma} \left[\int h_{-1N}dJ_{h_{-1N}}\right] \int \theta dF_N(\theta) + \frac{1}{w\gamma} [\gamma - 1 - \lambda + s\lambda] \int \theta dF_N(\theta)
\]

As I have shown above, the problem of traders and non-traders in the bond market are very similar to the case with no exogenous segmentation in the asset market. The same logic from the case with endogenous asset market segmentation applies and asset market
traders and non-traders solve optimization problems similar to the problem in the previous sections. The households’ decisions are only linked through the market clearing conditions and prices. We have a partial block recursive equilibrium in which the distributions in the economy affects households’ decision through prices. Households do not take in to account the distribution of asset holdings among traders and non-traders. The main results in the previous sections are robust to adding exogenously segmented asset market.
References


FDIC, Federal Deposit Insurance Corporation (2009) “National Survey of Unbanked and Underbanked Households.” 2


Menzio, Guido, Shouyong Shi, and Hongfei Sun (2011) “A Monetary Theory with Non-Degenerate Distributions.” Working paper. 4, 5, 13, 28


Sun, Hongfei (2012) “Monetary and Fiscal Policies In a Heterogeneous-Agent Economy.” Working Paper. 4, 5, 11, 12, 13, 14, 15, 28, 35


